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A NOTE ON GLOBAL TOTAL DOMINATION IN GRAPHS

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ABSTRACT

A dominating set S of a graph G is a global total dominating set if S is both a global dominating set and a total dominating set. The global total domination number $\gamma_{gt}(G)$ is the minimum cardinality of a global total dominating set of G. In this paper we discuss some results on global total domination number.

Keywords: global domination, total domination, global total domination, global total domination number.

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1. INTRODUCTION

All graphs under our consideration are finite, undirected, without loops, multiple edges and isolated vertices. Terms not defined here are used in the sense of Harary [1]. Let $G=(V,E)$ be a graph. A vertex in a graph G dominates itself and its neighbors. A set of vertices S in a graph G is a dominating set (DS), if each vertex of G is dominated by some vertices of S. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. The theory of domination is outlined in two books by Haynes, Hedetniemi and Slater [2,3].

A total dominating set (TDS) of a graph G with no isolated vertex is a set S of vertices of G such that every vertex is adjacent to a vertex in S. Every graph without isolated vertices has a TDS, since $S=V(G)$ is such a set. The total domination number $\gamma_t(G)$ of G is the minimum cardinality of a TDS. Total domination in graphs was introduced by Cockayne, Dawes, Hedetniemi and Slater [4].

A global dominating set (GDS) of G is a set of vertices that dominates both G and the complement graph $\overline{G}$. The global domination number $\gamma_g(G)$ of G is the minimum cardinality of a GDS. Global domination was introduced by Sampathkumar [5].

A total global dominating set (TGDS) of G is a total dominating set of both G and $\overline{G}$. The total global domination number $\gamma_{tg}(G)$ of G is the minimum cardinality of a TGDS. For this we refer the reader to [6].

We define the new concept namely **Global Total Dominating Set** as follows:

A global total dominating set (GTDS) of a graph G is a set S of vertices of G such that S is both GDS and TDS. The global total domination number $\gamma_{gt}(G)$ of G is the minimum cardinality of a GTDS.

We note that $\gamma(G)$ and $\gamma_g(G)$ are defined for any G. $\gamma_{tg}(G)$ is only defined for G with $\delta(G)\geq 1$ and $\delta(\overline{G})\geq 1$. $\gamma_t(G)$ and $\gamma_{gt}(G)$ are only defined for G with $\delta(G)\geq 1$, where $\delta(G)$ is the minimum degree of G.

Theorem1.1 For any graph G of order n, $2\leq\gamma_{gt}\leq n$.

Proof: A global total dominating set needs at least two vertices and so $\gamma_{gt}\geq 2$. The set of all vertices of G is clearly a GTDS of G so that $\gamma_{gt}\leq n$ .Thus $2\leq\gamma_{gt}\leq n$.

Remark 1.2 The bounds in Theorem1.1 are sharp. For the complete graph $K_n\,(n\geq 2), \gamma_{gt}(K_n)=n.$ For the complete bipartite graph $K_{m,n}, \gamma_{gt}(K_{m,n})=2.$ Thus $K_n\,(n\geq 2)$ has the largest possible GTD number n and the complete bipartite graphs have the smallest global total domination number.

Theorem 1.3 For any positive integers m,n, $\gamma_{gt}(K_{m,n})=2$.

Proof: Let G be a complete bipartite graph with partitions V_1 and V_2. Let $u\in V_1$ and $v\in V_2$. Since G is a bipartite graph, each vertex in one partition can dominate all vertices in the other partition. But in $\overline{G}$ it will dominate all vertices in it's own partition. Hence it is enough to have two vertices to dominate all vertices in both G and $\overline{G}$. Thus $S=\{u,v\}$ is both global and total dominating set in G.
Hence $\gamma_{gt}(K_{m,n})=2$.

2. Bounds for GTDS in graphs

Theorem 2.1 Let G be a connected graph, then $\gamma_{gt}(G) \geq \left\lceil \frac{n}{2\Delta-1} \right\rceil$.

Proof: Let $S \subseteq V(G)$ be a GTDS in G. Every vertex in S dominates at most $\Delta(G)-1$ vertices of $V(G)-S$ and dominates at most $\Delta(G)$ vertices in S.

Hence $|S|(\Delta-1)+|S|\Delta \geq n$.

$\Rightarrow |S|(2\Delta-1) \geq n$.

$\Rightarrow |S| \geq \frac{n}{2\Delta-1}$.

Since, S is an arbitrary global total dominating set, $\gamma_{gt}(G) \geq \left\lceil \frac{n}{2\Delta-1} \right\rceil$.

Note: If G is a complete bipartite graph with bipartition X, Y and $|X|=|Y|$, then $\gamma_{gt}(G) = \left\lceil \frac{n}{2\Delta-1} \right\rceil$. So the above bound is sharp.

Theorem 2.2 Let G be a connected graph, then $\gamma_{gt}(G) \leq n-\Delta+\delta$.

Proof: Let S be any γ_{gt}-set of G. Every vertex in S dominates at least one vertex in S and at least one vertex in S dominates at least $\Delta-\delta$ vertices in $V(G)-S$.

Hence $|S| + (\Delta-\delta) \leq n$.

$\Rightarrow |S| \leq n-\Delta+\delta$.

Since, S is an arbitrary global total dominating set, $\gamma_{gt}(G) \leq n-\Delta+\delta$.

Theorem 2.3 Let G be a graph of order $n \geq 3$. Then $\gamma_{gt}(G) = n-1$ if and only if $G \cong K_n - e$.

Proof: We first prove the sufficiency part. Let $G \cong K_n - e$ where $e = uv \in E(K_n)$. So $uv \notin E(G)$ and hence $uv \in E(\overline{G})$ and also $\overline{G}$ contains $n-2$ isolated vertices. Hence every GTDS of G must contain all vertices of $V(G)-\{u,v\}$ and at least one of u and v. Thus $\gamma_{gt}(G) \geq n-1$ --------------------(1)

Since $V(G)-\{u\}$ is a GTDS of G, it follows that $\gamma_{gt}(G) \leq n-1$ -----------------(2)

Thus by (1) and (2) $\gamma_{gt}(G) = n-1$.

Now we prove necessity.

Assume $\gamma_{gt}(G) = n-1$. To prove $G \cong K_n - e$. We know that $\gamma_{gt}(K_n) = n$. We proved that $\gamma_{gt}(K_n - e) = n-1$. Since $\gamma_{gt}(G) = n-1, G \cong K_n - e$.

Theorem 2.4 Let G be a graph with no isolated vertices. Then

$\gamma_{gt} G = n \Leftrightarrow G \cong K_n \ or \ mK_2$.

Proof: The proof of sufficiency is obvious.

To prove the necessity. Assume that $\gamma_{gt}(G) = n$.

Case 1 : G is connected.

Suppose $G \ncong K_n, n \geq 3$. Then there exists a vertex $v \in V$ such that $\deg(v) < n-1$. Then $V - \{v\}$is a GTDS, which is a contradiction. Thus $G \cong K_n$.

Case 2 : G is disconnected.

Suppose there exists $v \in V$ such that $\deg(v) \geq 2$.

Then $V - \{v\}$is a GTDS, which is a contradiction.

Therefore $\deg(v) = 1 \ \forall \ v \in V$. Thus $G \cong mK_2$.

Definition: 2.5 The greatest distance between any two vertices of a connected graph G is called the **diameter** of G and is denoted by $d(G)$.

Theorem 2.6 Let G be a graph with $1 \leq d(G) \leq 2$, then $\gamma_{gt}(G) \leq \delta(G) + 1$.

Proof: Let x be a vertex of minimum degree in G. Since $1 \leq d(G) \leq 2$, then $N(x)$ is a dominating set for G. Now $\{x\} \cup N(x)$ is a dominating set for $\overline{G}$ and also a total dominating set for G. Thus we have $S = \{x\} \cup N(x)$ is a global total dominating set for G and $|S| = \delta(G) + 1$.

Hence $\gamma_{gt}(G) \leq \delta(G) + 1$.

Remark 2.7 If v is a support vertex of a graph G, then v is in every $\gamma_{gt}(G)$-set.

Definition:2.8 The **degree** of a vertex v denoted by $d_G(v)$ is the number of edges incident with the vertex v. A **leaf** of a tree T is a vertex of degree one, while a **support vertex** of T is a vertex adjacent to a leaf.

Theorem 2.9 If T is a tree of order $n \geq 3$, then $\gamma_{gt}(T) \leq n - l + 1$. Moreover the equality holds if and only if T is a star.

Proof: Let T be a tree of order $n \geq 3$. Let S be any γ_{gt}-set. By Remark 2.7, S contains every support vertices of T. Since T has at most $n - l$ support vertices, then $|S| \leq n - l + 1$.

For the moreover part, if T is a star then by Theorem 1.3, $\gamma_{gt}(T) = 2 = n - l + 1$. Conversely let T be a tree with $\gamma_{gt}(T) = n - l + 1$. We show that T is a star.

Let S be a $\gamma_{gt}(T)$-set with size $|S| = n - l + 1$ and let v be a vertex of T with $\deg(v) = n - l$. If $v \notin S$ then $d_G(v) = 0$ which is impossible, so $v \in S$. Since $|S| = n - l + 1$, each vertex of $N(v)$ is an end vertex. Hence T is a star.

Theorem 2.10 Let G be a graph of order n and size m, $\delta(G) \geq 1$.

Then $\gamma_{gt}(G) \geq \frac{n}{2} - m$.

Proof: Let S be any γ_{gt}-set of G. Consider $A = \langle V - S \rangle$ and $B = \langle S \rangle$. Let n_1 and n_2 be the order of A and B respectively. Also m_1 and m_2 be the size of A and B respectively.

Thus $m_1 = \frac{1}{2} \sum_{v \in V-S} \deg_A(v) \geq \frac{1}{2}\left(n - \gamma_{gt}(G)\right)$ and

$m_2 = \frac{1}{2} \sum_{v \in S} \deg_B(v) \geq \frac{1}{2}\gamma_{gt}(G)$.

Let m_3 denote the number of edges between S and $V - S$. Since S is a γ_{gt}-set, and so S is a total dominating set every vertex is adjacent to at least one vertex in S. Thus $m_3 \geq \gamma_{gt}(G)$.

Hence

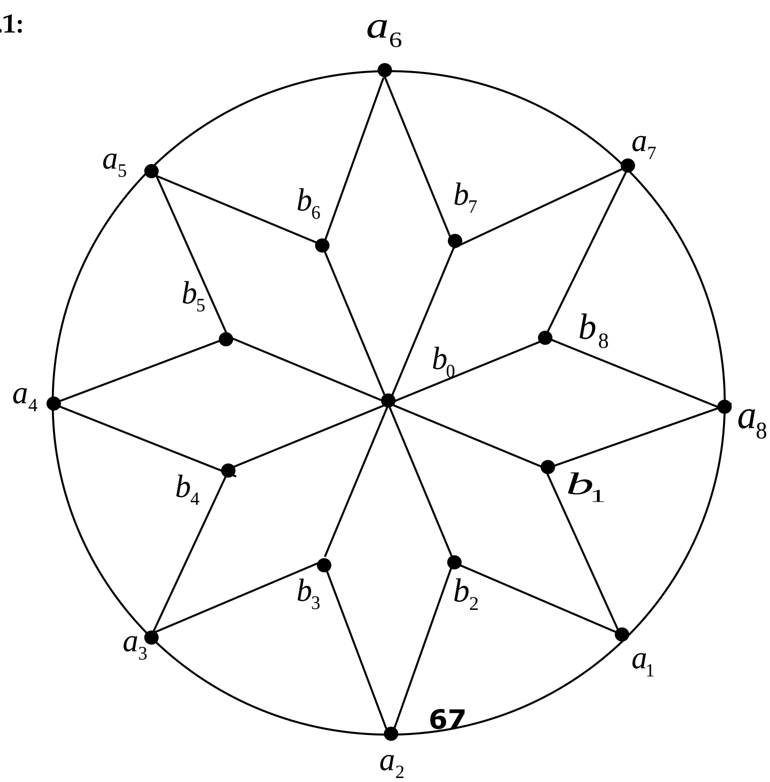
$$m = m_1 + m_2 + m_3$$
$$\geq \frac{1}{2}\left(n - \gamma_{gt}(G)\right) + \frac{1}{2}\gamma_{gt}(G) + \gamma_{gt}(G).$$
$$\Rightarrow m \geq \frac{1}{2}n - \frac{1}{2}\gamma_{gt}(G) + \gamma_{gt}(G).$$

Which implies that $\gamma_{gt}(G) \geq \frac{n}{2} - m$.

3. Lotus Inside Circle: [7]

The graph lotus inside circle is denoted by LIC_n, $n \geq 3$ and is defined as follows. Let S_n be the star graph with vertices $b_0, b_1, \ldots, b_n$ whose center is b_0. Let C_n be the cycle of length n whose vertices are $a_1, a_2, \ldots a_n$. We join a_{i+1} with b_i & b_{i+1} for each $i \geq 1$ and join a_1 with b_1 & b_r.

Example 3.1:



Theorem 3.2 For $n \geq 4$, $\gamma_{gt}\left(LIC_n\right) = \left\lceil \frac{n}{2} \right\rceil + 1.$

Proof:

Case 1: n is even. Consider the set $S = \{b_0, b_1, b_3, b_5, ..., b_{n-3}, b_{n-1}\}$. It is easy to see that S is a total dominating set for LIC_n and also b_0 and b_1 dominates $a_1, a_2, ..., a_n$ and $b_2, b_3, ..., b_n$ respectively in $\overline{LIC_n}$ and hence S is a GTDS for LIC_n.

Thus $\gamma_{gt}\left(LIC_n\right) \leq |S| = \left\lceil \frac{n}{2} \right\rceil + 1$.

Let $T \subset V\left(LIC_n\right), |T| \leq \left\lceil \frac{n}{2} \right\rceil$ and T be GTDS for LIC_n.

We split into three cases.

Subcase 1.1:

Suppose $b_0 \in T$ and $\left\lceil \frac{n}{2} \right\rceil - 1$ remained vertices of T be the vertices of star graph. Due to the structure of the graph LIC_n, b_0 dominates $b_1, b_2, ..., b_n$. In this case $T - b_0$ must dominate $a_1, a_2, ..., a_n$. But $T - b_0$ dominate at most $n-2$ vertices of the cycle C_n. So at least two vertices of C_n that any vertices of T cannot dominate them, which is a contradiction.

Subcase 1.2:

Let $b_0, b_1 \in T$ and $\left\lceil \frac{n}{2} \right\rceil - 2$ remained vertices of T be the vertices of cycle C_n. Due to the structure of the graph LIC_n, b_0 dominates $b_1, b_2, ..., b_n$ and b_1 dominates a_1 and a_n. In this case T must dominate $n-2$ vertices of the cycle C_n with $\left\lceil \frac{n}{2} \right\rceil - 2$ vertices of C_n. Here at least two vertices of cycle C_n are not dominated by any vertices of T which is a contradiction.

Subcase 1.3:

Let $b_0 \notin T$, without loss of generality suppose that $b_1, a_1 \in T$. Then b_1 dominates itself and the vertices $b_0, a_1 \& a_n$ and a_1 dominates itself and the vertices b_1, b_2, a_2, a_n. So the remained $\left\lceil \frac{n}{2} \right\rceil - 2$ vertices of T dominates all other vertices in

$V(LIC_n)$. Since at least $\left\lceil \frac{n}{2} \right\rceil$ of vertices in $V(LIC_n)$ are not dominated by vertices in T. Thus we have a contradiction.

Hence $\gamma_{gt}(LIC_n) \geq \left\lceil \frac{n}{2} \right\rceil + 1.$

This implies that $\gamma_{gt}(LIC_n) = \left\lceil \frac{n}{2} \right\rceil + 1.$

Case 2 : n is odd.

It is easy to verify that the set of vertices $S_1 = \{b_0, b_1, b_3, \ldots, b_{n-2}, b_n\}$ is both total and global dominating set for LIC_n and hence S_1 is GTDS for LIC_n. Therefore $\gamma_{gt}(LIC_n) \leq |S_1| = \left\lceil \frac{n}{2} \right\rceil + 1$. So it is enough to prove that $\gamma_{gt}(LIC_n) \geq \left\lceil \frac{n}{2} \right\rceil + 1$. Let $T \subset V(LIC_n)$, $|T| \leq \left\lceil \frac{n}{2} \right\rceil$ and T be GTDS of LIC_n.

We split into three cases.

Subcase 2.1

Let $b_0 \in T$ and $\left\lceil \frac{n}{2} \right\rceil - 1$ remained vertices of T be the vertices of star graph S_n. Due to the structure of the graph LIC_n, b_0 dominates $b_1, b_2, \ldots, b_n$. So T must dominate all vertices of C_n. $T - b_0$ dominate at most $n-1$ vertices of cycle C_n. So at least one vertex of C_n that any vertices of T cannot dominate them, which is a contradiction.

Subcase2. 2

Let $b_0, b_1 \in T$ and $\left\lceil \frac{n}{2} \right\rceil - 2$ remained vertices of T be the vertices of cycle C_n. Due to the structure of the graph LIC_n, b_0 dominates $b_1, b_2, \ldots, b_n$ and b_1 dominates a_1 and a_n. So the remaining $n-2$ vertices in C_n are must dominated by $\left\lceil \frac{n}{2} \right\rceil - 2$ vertices of T. Here at least one vertex of C_n is not dominated by any vertex of T, which is a contradiction.

Subcase 2.3

Let $b_0 \notin T$, without loss of generality suppose that $b_1, a_1 \in T$. Then b_1 dominates itself and the vertices $b_0, a_1 \& a_n$ and a_1 dominates itself and the vertices b_1, b_2, a_2, a_n. In this case, the remaining $\left\lceil \frac{n}{2} \right\rceil - 2$ vertices of T must dominates all other vertices in $V\left(LIC_n\right)$. Here at least $\left\lceil \frac{n}{2} \right\rceil - 3$ vertices in $V\left(LIC_n\right)$ are not dominated by vertices in T . Thus we have a contradiction.

Hence $\gamma_{gt}\left(LIC_n\right) \geq \left\lceil \frac{n}{2} \right\rceil + 1$.

This implies that $\gamma_{gt}\left(LIC_n\right) = \left\lceil \frac{n}{2} \right\rceil + 1$.

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